

6.4 - The Determinant of a Square Matrix

A determinant is a real number associated with every square matrix. I have yet to find a good English definition for what a determinant is. Everything I can find either defines it in terms of a mathematical formula or suggests some of the uses of it. There's even a definition of determinant that defines it in terms of itself.

The determinant of a square matrix A is denoted by " $\det A$ " or $|A|$. Now, that last one looks like the absolute value of A , but you will have to apply context. If the vertical lines are around a matrix, it means determinant.

The line below shows the two ways to write a determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = \det \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Determinant of a 2×2 Matrix

The determinant of a 2×2 matrix is found much like a [pivot](#) operation. It is the product of the elements on the main diagonal minus the product of the elements off the main diagonal.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Properties of Determinants

- The determinant is a real number, it is not a matrix.
- The determinant can be a negative number.
- It is not associated with absolute value at all except that they both use vertical lines.
- The determinant only exists for square matrices (2×2, 3×3, ... n×n). The determinant of a 1×1 matrix is that single value in the determinant.
- The inverse of a matrix will exist only if the determinant is not zero.

Expansion using Minors and Cofactors

The definition of determinant that we have so far is only for a 2×2 matrix. There is a shortcut for a 3×3 matrix, but I firmly believe you should learn the way that will work for all sizes, not just a special case for a 3×3 matrix.

The method is called expansion using minors and cofactors. Before we can use them, we need to define them.

Minors

A minor for any element is the determinant that results when the row and column that element are in are deleted.

The notation M_{ij} is used to stand for the minor of the element in row i and column j . So M_{21} would mean the minor for the element in row 2, column 1.

Consider the 3×3 determinant shown below. I've included headers so that you can keep the rows and columns straight, but you wouldn't normally include those. We're going to find some of the minors.

$$\begin{array}{c|ccc|} & \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} & \\ \mathbf{R_1} & 1 & 3 & 2 & \\ \mathbf{R_2} & 4 & 1 & 3 & \\ \mathbf{R_3} & 2 & 5 & 2 & \end{array}$$

Finding the Minor for R_2C_1

The minor is the determinant that remains when you delete the row and column of the element you're trying to find the minor for. That means we should delete row 2 and column 1 and then find the determinant.

$$\begin{array}{c|cc|} & \mathbf{C_2} & \mathbf{C_3} & \\ \mathbf{R_1} & 3 & 2 & \\ \mathbf{R_3} & 5 & 2 & \end{array} = 3(2) - 5(2) = 6 - 10 = -4$$

As you can see, the minor for row 2 and column 1 is $M_{21} = -4$.

Let's try another one.

Finding the Minor for R_3C_2

This time, we would delete row 3 and column 2.

$$\begin{array}{c|cc|} & \mathbf{C_1} & \mathbf{C_3} & \\ \mathbf{R_1} & 1 & 2 & \\ \mathbf{R_2} & 4 & 3 & \end{array} = 1(3) - 4(2) = 3 - 8 = -5$$

So the minor for row 3, column 2 is $M_{32} = -5$.

Matrix of Minors

When you're just trying to find the determinant of a matrix, this is overkill. But there is one extremely useful application for it and it will give us practice finding minors.

The matrix of minors is the square matrix where each element is the minor for the number in that position.

Here is a generic matrix of minors for a 3×3 determinant.

$$\begin{array}{c} \mathbf{R_1} \\ \mathbf{R_2} \\ \mathbf{R_3} \end{array} \begin{bmatrix} \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} \\ M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Let's find the matrix of minors for our original determinant. Here is the determinant.

$$\begin{array}{c} \mathbf{R_1} \\ \mathbf{R_2} \\ \mathbf{R_3} \end{array} \begin{vmatrix} \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} \\ 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$$

Here is the work to find each minor in the matrix of minors.

	C₁	C₂	C₃
R₁	$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix}$ = 2 - 15 = -13	$\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix}$ = 8 - 6 = 2	$\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$ = 20 - 2 = 18
R₂	$\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix}$ = 6 - 10 = -4	$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$ = 2 - 4 = -2	$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ = 5 - 6 = -1
R₃	$\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$ = 9 - 2 = 7	$\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ = 3 - 8 = -5	$\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$ = 1 - 12 = -11

Finally, here is the matrix of minors. Again, you don't need to put the labels for the row and columns on there, but it may help you.

$$\begin{array}{c} \mathbf{R_1} \\ \mathbf{R_2} \end{array} \begin{bmatrix} \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} \\ -13 & 2 & 18 \\ -4 & -2 & -1 \end{bmatrix}$$

$$R_3 \begin{bmatrix} 7 & -5 & -11 \end{bmatrix}$$

Cofactors

A cofactor for any element is either the minor or the opposite of the minor, depending on where the element is in the original determinant. If the row and column of the element add up to be an even number, then the cofactor is the same as the minor. If the row and column of the element add up to be an odd number, then the cofactor is the opposite of the minor.

Ooh - did you get that? Odd changes signs, even is the same sign. Deja Vu. We've been talking about that ever since section 3.2 on polynomials.

Sign Chart

Rather than adding up the row and column of the element to see whether it is odd or even, many people prefer to use a sign chart. A sign chart is either a + or - for each element in the matrix. The first element (row 1, column 1) is always a + and it alternates from there.

Note: The + does not mean positive and the - negative. The + means the same sign as the minor and the - means the opposite of the minor. Think of it addition and subtraction rather than positive or negative.

Here is the sign chart for a 2x2 determinant.

$$\begin{array}{c|cc} & C_1 & C_2 \\ R_1 & + & - \\ R_2 & - & + \end{array}$$

Here is the sign chart for a 3x3 determinant.

$$\begin{array}{c|ccc} & C_1 & C_2 & C_3 \\ R_1 & + & - & + \\ R_2 & - & + & - \\ R_3 & + & - & + \end{array}$$

Matrix of Cofactors

Again, if all you're trying to do is find the determinant, you do not need to go through this much work.

The matrix of cofactors is the matrix found by replacing each element of a matrix by its cofactor. This is the matrix of minors with the signs changed on the elements in the - positions.

$$C_1 \quad C_2 \quad C_3$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} -13 & -2 & 18 \\ 4 & -2 & 1 \\ 7 & 5 & -11 \end{bmatrix}$$

Expanding to Find the Determinant

Here are the steps to go through to find the determinant.

1. Pick any row or column in the matrix. It does not matter which row or which column you use, the answer will be the same for any row. There are some rows or columns that are easier than others, but we'll get to that later.
2. Multiply every element in that row or column by its cofactor and add. The result is the determinant.

Let's expand our matrix along the first row.

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$$

From the sign chart, we see that 1 is in a positive position, 3 is in a negative position and 2 is in a positive position. By putting the + or - in front of the element, it takes care of the sign adjustment when going from the minor to the cofactor.

$$+ 1 \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} - 3 \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= 1 (2 - 15) - 3 (8 - 6) + 2 (20 - 2)$$

$$= 1 (-13) - 3 (2) + 2 (18)$$

$$= -13 - 6 + 36$$

$$= 17$$

The determinant of this matrix is 17.

As I said earlier, it doesn't really matter which row or column you use.

Let's try it again, but this time expand on the second columns. As an effort to save time, the minors for that column (from the matrix of minors) were 2, -2, and -5. The original elements were 3, 1, and 5. The 3 and 5 are in negative positions.

$$\text{determinant} = - 3 (2) + 1 (- 2) - 5 (- 5) = -6 - 2 + 25 = 17$$

Expand on any row or any column, you'll get 17.

However, you can't do diagonals. If we try the main diagonal, you get

$$+ 1 (-13) + 1 (-2) + 2 (-11) = -13 - 2 - 22 = -37$$

Some rows or columns are better than others

1. Pick the row or column with the most zeros in it.
 Since each minor or cofactor is multiplied by the element in the matrix, picking a row or column with lots of zeros in it means that you will be multiplying by a lot of zeros. Multiplying by zero doesn't take very long at all. In fact, if the element is zero, you don't need to even find the minor or cofactor.
2. Pick the row or column with the largest numbers (or variables) in it.
 The elements in the row or column that you expand along are not used to find the minors. The only place that they are multiplied is once, in the expansion. If you pick the row or column with the smallest numbers, then every minor will be the product of larger numbers.
 If you pick a row or column that has variables in it, then you will only have to multiply by the variables once, during the expansion.

Inverse of a Matrix (revisited)

Let's consider our original determinant as a matrix this time.

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

Find the **matrix of minors** as explained above.

$$\begin{bmatrix} -13 & 2 & 18 \\ -4 & -2 & -1 \\ 7 & -5 & -11 \end{bmatrix}$$

Turn it into a **matrix of cofactors** by changing the signs on the appropriate elements based on the sign chart.

$$\begin{bmatrix} -13 & -2 & 18 \\ 4 & -2 & 1 \\ 7 & 5 & -11 \end{bmatrix}$$

Find the **adjoint** by transposing the matrix of cofactors.

To transpose a matrix, you switch the rows and columns. That is, the rows become columns and the columns become rows. The Transpose of a matrix can be found using the TI-82 or TI-83 calculator by entering the name of the matrix and then choosing Matrix, Math, and then option 2, a superscripted T, like $[A]^T$.

$$\begin{bmatrix} -13 & 4 & 7 \\ -2 & -2 & 5 \\ 18 & 1 & -11 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 5 \\ 18 & 1 & -11 \end{bmatrix}$$

Finally divide the adjoint of the matrix by the determinant of the matrix. In this problem, the determinant is 17, so we'll divide every element by 17. The resulting matrix is the **inverse** of the original matrix.

$$\begin{bmatrix} -13/17 & 4/17 & 7/17 \\ -2/17 & -2/17 & 5/17 \\ 18/17 & 1/17 & -11/17 \end{bmatrix}$$

The inverse of a matrix is found by dividing the adjoint of the matrix by the determinant of the matrix. Don't try that on your calculator since the calculator won't let you divide a matrix by a scalar. You will have to multiply by the inverse of the determinant instead.

If you check it with your calculator, you can verify that the inverse actually is the adjoint divided by the determinant.

Since the inverse is the adjoint divided by the determinant, we can see why the inverse doesn't exist if the determinant is zero. That would cause division by zero, which is undefined.

Larger Order Determinants

Let's find the determinant of a 4x4 system.

	C₁	C₂	C₃	C₄
R₁	3	2	0	1
R₂	4	0	1	2
R₃	3	0	2	1
R₄	9	2	3	1

Pick the row or column with the most zeros in it. In this case, that is the second column.

For each element in the original matrix, its minor will be a 3x3 determinant. We'll have to expand each of those by using three 2x2 determinants.

This is why we want to expand along the second column. The minors are multiplied by their elements, so if the element in the original matrix is 0, it doesn't really matter what the minor is and we can save a lot of time by not having to find it. In the second column, you won't need to find two of the minors because their corresponding element in the second column is zero.

$$-2 \begin{vmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 9 & 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

We could actually fill in those middle two minors, but since they're multiplied by 0, it doesn't really matter what they are. In fact, you could just as easily skip them.

Now, there are two 3x3 determinants left to find.

In the first 3x3 determinant, there are no zeros, so pick the row or column with the largest numbers. That would be column 1, so expand along the first column.

Notice the 4 is in a positive position. The sign charts begin over with each new determinant. The position of the number in the original matrix does not matter, only its position in the current matrix.

$$\begin{vmatrix} 4 & 1 & 2 \\ 3 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = +4 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 9 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 4(2 - 3) - 3(1 - 6) + 9(1 - 4) = 4(-1) - 3(-5) + 9(-3) = -4 + 15 - 27 = -16$$

Consider the other 3x3 matrix. In this one, there is a 0 in the row 1 and column 2. Either one of those would be a good pick for expansion, but since row 1 has slightly larger numbers, we'll expand along the first row.

$$\begin{vmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = +3 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} + 1 \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 3(1 - 4) - 0(\text{doesn't matter}) + 1(8 - 3) = 3(-3) + 1(5) = -9 + 5 = -4$$

When you go to find the determinant, remember that there were elements from the original 4x4 matrix that were times each of those 3x3 determinants. The first one was -2 and the second one was +2.

$$\text{Determinant} = -2(-16) + 2(-4) = 32 - 8 = 24$$

Worst case scenario

To find a 3x3 determinant with no zeros, you have to find three 2x2 determinants.

To find a 4x4 determinant with no zeros, you have to find four 3x3 determinants, each of which then becomes three 2x2 determinants for a total of twelve 2x2 determinants.

To find a 5x5 determinant with no zeros, you have to find five 4x4 determinants, each of which then becomes four 3x3 determinants, each of those becoming three 2x2 determinants for a total of sixty 2x2 determinants.

Using the Calculator

After that last problem, you've got to be asking yourself if there isn't an easier way. Well, yes,

there is, as long as the determinant doesn't have any variables in it. You can use the calculator.

The notation that the TI-82 or TI-83 calculator uses is the Det A notation. So, after entering the matrix into one of the available matrices on the calculator, enter DET by going Matrix, Math, and choosing option 1. Then put in the name of the matrix that you're using.

You don't need to use parentheses (unless you have a TI-83), but you can if you want to find the determinant of a product " $\det([A][B])$ " or the determinant of a transpose " $\det([A]^T)$ " as opposed to the transpose of the determinant " $(\det[A])^T$ ". By the way, the calculator won't find the transpose of a determinant because the determinant is a scalar (real number) and the calculator only knows how to find the transpose of a matrix. The transpose of a scalar is that scalar.

Triangular Matrices

You're really going to like finding determinants of these matrices.

Upper Triangular Matrix

A matrix in which all the non-zero elements are either on or above the main diagonal.

That is, all the non-zero values are in the upper triangle. Everything below the diagonal is a zero.

Lower Triangular Matrix

A matrix in which all the non-zero elements are either on or below the main diagonal.

That is, all the non-zero values are in the lower triangle. Everything above the diagonal is zero.

Diagonal Matrix

A matrix in which all the non-zero elements are on the main diagonal. Everything off the main diagonal is a zero.

The determinant of a triangular matrix or a diagonal matrix is the product of the elements on the main diagonal.

Elementary Row Operations

There were three elementary row operations that could be performed that would return an equivalent system. With determinants, since the determinant of a transpose is the same as the determinant of the matrix, the elementary row operations can also be applied to columns.

By performing row-reduction (using pivoting on a 1 if you like), you can place a matrix into triangular form. Once it's in triangular form, then all you have to do is multiply by the elements on the main diagonal and you have the determinant.

Let's look at each of the three elementary row operations.

1. If you interchange two rows or two columns in a determinant, the resulting determinant will differ only in sign. That is, if you swap rows or columns, the resulting determinant is the opposite of the original determinant.
2. If you multiply a row or column by a non-zero constant, the determinant is multiplied by that same non-zero constant.

3. If you multiply a row or column by a non-zero constant and add it to another row or column, replacing that row or column, there is no change in the determinant.

That last operation is equivalent to pivoting on a one!

Warning, if your pivot is a number other than one, then you are multiplying each row that you change by the pivot element. So, if you pivot on a 3 and you change two rows, then the resulting determinant will be $3 \cdot 3 = 9$ times as great as the original determinant.

As long as you pivot on a one, you'll be okay.

You do not have to place the matrix into reduced row-echelon form or even row-echelon form. You are free to stop the reduction at any point and expand using minors and cofactors. What I suggest is pivot where there is a one, and then expand.

Determinants that are Zero

The determinant of a matrix will be zero if

1. An entire row is zero.
2. Two rows or columns are equal.
3. A row or column is a constant multiple of another row or column.

Remember, that a matrix is invertible, non-singular, if and only if the determinant is not zero. So, if the determinant is zero, the matrix is singular and does not have an inverse.