THE EXPRESSION OF A TENSOR OR A POLYADIC AS A SUM OF PRODUCTS

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1. Addition and Multiplication.

Tensors are added by adding corresponding components. The product of a covariant tensor $A_{i_1 \cdots i_p}$ of order $p$ into a covariant tensor $B_{i_{p+1} \cdots i_{p+q}}$ of order $q$ is defined by writing

$$A_{i_1 \cdots i_p} B_{i_{p+1} \cdots i_{p+q}} = C_{i_1 \cdots i_{p+q}}$$

where the product $C_{i_1 \cdots i_{p+q}}$ is a covariant tensor of order $p+q$. When no confusion results indices may be omitted giving

$$AB = C$$

equivalent to the $n^{p+q}$ equations (1). Boldface type is convenient for indicating that the letters do not denote merely numbers or scalars. Products of contravariant and of mixed tensors may be similarly defined.

A partial statement of the problem to be considered is as follows: to find under what conditions a given tensor can be expressed as a sum of products of assigned form. A more general statement of the problem will be given below.

2. Polyadic form of a tensor.

Any covariant tensor $A_{i_1 \cdots i_p}$ can be expressed as the sum of a finite number of tensors each of which is the product of $p$ covariant vectors,

$$A_{i_1 \cdots i_p} = \sum_{j=h} a_{i_1 j} a_{i_2 j} \cdots a_{i_p j}$$

where $a_{i_1 j}, a_{i_2 j}, \text{etc.}$, are a set of $hp$ covariant vectors. When the indices $i_1 \cdots i_p$ can be omitted this may be written

$$A = \sum_{j=h} a_{i j} a_{i_2 j} \cdots a_{i_p j}.$$  \hspace{1cm} (2')

The right member is now identical in appearance with a Gibbs
polyadic, defined as a sum of polyads. A covariant tensor developed as in (2) is \textit{in polyadic form}. Contravariant and mixed tensors may be similarly expanded.

A case of the present problem is to find the minimum value of \( h \) for assigned values of \( n \) and of \( p \), with or without further restrictions.

3. \textbf{Development in terms of basis vectors.}

Let \( e_i, e_m, e_2, e_m, \ldots, e_n, e_m \) denote a set of \( n \) covariant vectors each of which, at the point considered and with the variables employed, has components 1 or 0 according as its two subscripts are equal or not. We then have identically

\[
A_{i_1 \ldots i_p} = A_{a_1 \ldots a_p} e_{a_1}, i_1 e_{a_2}, i_2 \cdots e_{a_p}, i_p
\]

where the occurrence of Greek indices twice on the right indicates summation from 1 to \( n \). When Roman indices can be omitted this becomes

\[
A = A_{a_1 \ldots a_p} e_{a_1} e_{a_2} \cdots e_{a_p}.
\]

The set of \( n \) vectors \( e_{a_m}, e_m \) defined above are \textit{basis vectors of the} \( m^{\text{th}} \) \textit{index}. The \( n^p \) products \( e_{a_1} e_{a_2} \cdots e_{a_p} \) obtained by assigning all possible sets of values to the subscripts constitute a set of \( n^p \) tensors in terms of which any covariant tensor \( A_{i_1 \ldots i_p} \) is linearly expressible. Any contravariant or any mixed tensor may be similarly developed.

4. \textbf{Differences between tensors and polyadics.}

A polyadic whose components are \( A_{i_1 \ldots i_p} \) may likewise be developed in the form (3). Like tensors, polyadics are added by adding corresponding components. Their (direct indeterminate) product obeys the rule (1). If therefore a given tensor can be expressed as a sum of products in a particular manner the same is true of that polyadic which possesses components numerically equal to the components of the tensor at the point considered and with the variables employed, and conversely. But a polyadic\(^1\)

\(^1\)I refer to the (affine) polyadics which, with their laws of multiplication, are defined in the author's paper \textit{A Theory of Ordered Determinants with Application to Polyadics}, this Journal, Vol. IV, No. 4, July 1925, pp. 212, 222. These polyadics differ from those of Gibbs only in the fact that the vectors employed have no metric, \textit{i.e.}, there is no dot product and no unit vector.
differs from a tensor in the following essential respect: a polyadic is subject to an arbitrary linear transformation on any assigned index thus,

\[ A'_{i_1 \cdots i_{m-1} i_m} = A_{i_1 \cdots i_{m-1} i_m} T_{a_{m}^2} \]  

a system of \( n^p \) equations in which the occurrence of the Greek index twice on the right indicates summation from 1 to \( n \); the \( T_{a_{m}^2} \) are a set of \( n^2 \) numbers whose determinant is not zero. I shall then say that \( A' \) is the transform of \( A \) on the \( m^{th} \) index by the transformation \( T \). A polyadic may be thus transformed on any or all indices. The transformation on one index need not be the same as that on any other index. A tensor on the other hand is transformed on every index by the particular transformations known as covariant or contravariant.

5. The invariant character of a particular formulation.

The possibility of formulating a tensor in a particular manner is interesting at least in so far as such possibility is an invariant property of the tensor. One way to establish the invariance is to show that the property in question depends on a system of equations which, by their very form, are invariant. In any case, however, the possibility of the formulation will depend on the vanishing, or it may be on the non-vanishing, of certain functions, or upon the vanishing of some and the non-vanishing of others, — excepting, obviously, such formulations as are possible for all tensors. To say that the formulation is possible is the same as saying that a certain set of equations has a solution, *i.e.*, the given tensor is equal to a required tensor of assigned form. The possibility of solution will be a property *invariant in the sense of tensor calculus* if, and only if, the determinants or other functions on which the solution depends are invariant in a quite different sense, namely *in that of the ordinary theory of quantics*.

Under the transformation (4), if a function of the \( A' \) is equal to the same function of the untransformed \( A \) multiplied by a power of the determinant of \( T \), I shall say that this function is an *invariant of the polyadic on the \( m^{th} \) index*. If it is an invariant

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on every index (separately) I shall say that it is a multiple invariant of the polyadic.\(^3\) Any property which persists when the polyadic is subject to arbitrary transformation on any or every index as in (4) will be called a multiply invariant property of the polyadic.

It is evident that, if a property of a polyadic is multiply invariant, it will be an invariant property (in the sense of tensor calculus) of any tensor whose components at the point in question and with the variables employed are numerically equal to those of the polyadic, whether the tensor be covariant, contravariant, or mixed. For if the property persists after an arbitrary linear transformation on each index it will persist when the transformation has on every index one of the particular forms known as covariant or contravariant.

For example, a dyadic \(A_{rs}\) can always be written as a sum of dyads thus,

\[
A_{rs} = \sum_{j=1}^{j=h} a_{j,r} b_{j,s}
\]

where the number of dyads \(h\) need not exceed \(n\). The same is true of tensors whether the indices \(r\) and \(s\) are upper or lower. It is well known that \(h\) can be less than \(n\) if, and only if, the determinant of the \(A_{rs}\) vanishes. But this determinant is a multiple invariant of the dyadic. If, therefore, \(h\) can be less than \(n\) for a given tensor (covariant, contravariant, or mixed) this property is invariant in the sense of tensor calculus.

If, on the other hand, a given property depends on a function of the polyadic which is merely invariant, without being multiply invariant, i.e., if the function is invariant when transformed on all indices simultaneously by the same \(T\) but is not invariant when transformed on an arbitrarily chosen single index, the property will be invariant for covariant tensors and for contravariant tensors, but not in general for mixed tensors.

For example a symmetric dyadic can be written

\[
A_{rs} = \sum_{j=1}^{j=h} a_{j,r} a_{j,s}
\]

\(^3\) A fuller discussion of invariants and other concomitants of polyadics is contained in the author's paper *A New Method in the Theory of Quantics*, this Journal, Vol. IV, No. 4, July 1925, p. 238.
and this is an invariant property of the dyadic, for symmetry evidently persists if the \( a_{j,r} \) and the \( a_{j,s} \) are simultaneously transformed by the same \( T \). It is not a multiply invariant property, for symmetry is destroyed if the \( a_{j,r} \) are transformed while the \( a_{j,s} \) are not. Accordingly symmetry is an invariant property of a covariant tensor \( A_{rs} \) or of a contravariant tensor \( \Lambda^{rs} \), but not of a mixed tensor \( A_{r}^{s} \) as is well known.

To illustrate further, take two dimensions, \( n = 2 \). The criterion that the dyadic (5) be symmetrical becomes

\[
A_{12} - A_{21} = 0
\]

that is

\[
\sum_{j=1}^{n} (a_{j,1} b_{j,2} - a_{j,2} b_{j,1}) = 0
\]

or more briefly

\[
\Sigma(ab) = 0
\]

where \((ab)\) denotes the space complement of \( a \) and \( b \) which, when \( n = 2 \), is the determinant of their components. The left member is an invariant of the dyadic for it is merely multiplied by the determinant of \( T \) when all vectors are transformed by \( T \). This is not true when prefactors only, or postfactors only, are transformed. Thus \( \Sigma(ab) \) is not a multiple invariant of the dyadic.

6. **Plan of the investigation and terminology employed.**

As above shown, any multiply invariant property of a polyadic corresponds to a property of tensors which is invariant in the sense of tensor calculus. (It is not true that, conversely, an invariant property of a tensor must yield an invariant property of a polyadic, because the transformations of a tensor are of a special sort.) The most interesting decompositions (with exception of those which presume symmetry or antisymmetry) appear to be such as are multiply invariant. The problem of decomposition of tensors into sums of products is thus (from a purely algebraic point of view) akin to the following problem: the determination of those properties of polyadics which are multiply invariant, and the discovery of the invariants, or other concomitants, on whose vanishing or non-vanishing the property depends.
It is furthermore evident that the concepts natural to the problem derive more or less from four sources, — tensor calculus, multiple algebra, quantics, and \( p \)-way determinants. The main concept is that of a matrix \( A_{i_1 \ldots i_p} \). The number of indices is called in this paper the \textit{order} of the polyadic and related tensors. This is not free from objection, since it corresponds to the \textit{class} of a \( p \)-way determinant.

In keeping with the notion of multiple invariance, any set of \( n \) basis vectors \( e_{am} \) or \( e_{am, i_m} \) defined in Art. 3, \textit{i.e.}, the basis vectors of the \( m \)th index, is regarded as a \textit{distinct} set, distinct that is from the basis vectors of any other index. When necessary this fact may be emphasized by an upper index, as \( e_{am}^{(m)} \).

In the polyadic development (2) any vector \( a_{mj} \) is a \textit{vector factor of the} \( m \)th \textit{index}, and is linearly expressible in terms of the basis vectors of the same index as \( a_{mj} = a_{am} e_{am} \). The totality of all possible vectors of this latter form constitutes the \textit{vector complex of the} \( m \)th \textit{index}.

Finally, the word \textit{vector} will in general be employed to denote a factor of a polyad. If the vector is covariant or contravariant in the sense of tensor calculus, the fact, if essential to the argument, will be so stated. Thus the basis vectors of a particular index become merely a set of \( n \) quantities, not commutative in multiplication, in terms of which all vectors of the \( m \)th index are linearly expressible.

7. \textbf{Rank of a} \( p \)-way \textbf{matrix}.

Consider the \( p \)-way matrix \( A_{i_1 \ldots i_p} \) of a polyadic or of a tensor. Let all indices be given fixed numerical values except a chosen index \( i_m \). Varying \( i_m \) from 1 to \( n \) yields \( n \) components called a \textit{file of the} \( m \)th \textit{index}. The \( n^{p-1} \) files of the \( m \)th index may be arranged as a two-way matrix having \( n \) rows and \( n^{p-1} \) columns. The columns are to stand in a definite order, called \textit{natural order}, defined as follows: let \( i_k \) be the first index of the series \( i_1, i_2, \ldots i_p \), which has a different (fixed) value in two chosen columns; that column \textit{precedes} in which \( i_k \) has the smaller value. Any (two-way) determinant of order \( q \) from this matrix will be called a \textit{minor} of \( A_{i_1 \ldots i_p} \) on the \( m \)th \textit{index}. 
For example, let $n=2$ and $p=3$. The four files of the first index form a matrix

$$
\begin{array}{cccc}
A_{111} & A_{112} & A_{121} & A_{122} \\
A_{211} & A_{212} & A_{221} & A_{222}
\end{array}
$$

whence there are six second order minors of the first index. In a similar manner there are six second order minors of each of the other indices. Only twelve of these eighteen minors are distinct.

If all minors of order $k+1$ on the $m^{th}$ index vanish, while at least one minor of order $k$ on this index does not vanish, the matrix $A_{i_1\ldots i_p}$ is of rank $k$ on the $m^{th}$ index.

8. Development according to rank on a chosen index.

Consider the polyadic form (2a) of a polyadic or of a tensor. The vector factors $a_{mj}$ of the $m^{th}$ index in general lie in an $n$-space, that is, any $n+1$ of them are linearly related. If any $k+1$ of them are linearly related these vectors may be said to lie in a $k$-space (although this phraseology is not wholly free from objection).

**Theorem I.** If the vector factors of the $m^{th}$ index lie in a $k$-space, the rank of $A_{i_1\ldots i_p}$ on the $m^{th}$ index is not greater than $k$.

**Proof.** The $k$ vectors $a_{mj}$ are linearly expressible in terms of $k$ properly chosen vectors of the $m^{th}$ complex. Let these be $b_{r,m}$ where $r=1,2,\ldots,k$. Substituting for the $a_{mj}, i_m$ in (2) their values in terms of the $b_{r,m}$ and collecting terms, the coefficient of each $b_{r,m}$ will be a polyadic of order $p-1$ which may be called $B_{r,i_1\ldots i_{m-1}i_{m+1}\ldots i_p}$. Thus (2) takes the form

$$
A_{i_1\ldots i_p} = \sum_{r=1}^{r=k} b_{r,m} B_{r,i_1\ldots i_{m-1}i_{m+1}\ldots i_p}
$$

which shows that, as a matrix of $n$ rows and $n^{p-1}$ columns, $A_{i_1\ldots i_p}$ is the product of two matrices which may be written so that the first has $k$ columns and the second $k$ rows. Hence the rank of $A_{i_1\ldots i_p}$ on the $m^{th}$ index cannot exceed $k$.

**Theorem II.** If the matrix $A_{i_1\ldots i_p}$ of a polyadic or a tensor is of rank $k_1$ on the index $i_1$, rank $k_2$ on the index $i_2$, etc., up to $k_p$ on the index $i_p$, $A$ can be developed in polyadic form in such a manner that,
simultaneously, the vector factors $a_{mj}$ of the $m^{th}$ index lie in a $k_m$-space for $m = 1, 2, \cdots, p$.

Proof. The right member of the identical relation (3a) may be regarded as the sum of $n^{p-1}$ vectors $A_{i_1 \cdots i_p} e_{a_1} \cdots e_{a_p}$ each multiplied into a polyad $e_{a_1} e_{a_2} \cdots e_{a_p}$. If the matrix $A_{i_1 \cdots i_p}$ is of rank $k_1$ on the first index these vectors (file vectors of the first index) may be expressed in terms of a properly chosen set of $k_1$ linearly independent vectors which may be called $a_{ij_1}$ where $j_1 = 1, 2, \cdots, k_1$. Substituting for the vectors $A_{i_1 \cdots i_p} e_{a_1}$ their values in terms of the $a_{ij_1}$ and collecting terms, the coefficient into which each $a_{ij_1}$ is multiplied will be a polyadic of order $p-1$ which may be called $B_{ij_1}$. Thus $A$, if of rank $k_1$ on $i_1$, can be developed as

$$A = \sum_{j_1 = 1}^{j_1 = k_1} a_{ij_1} B_{j_1}$$

which is the same as

$$A_{i_1 \cdots i_p} = \sum_{j_1 = 1}^{j_1 = k_1} a_{ij_1, i_1} B_{j_1, i_1, i_2 \cdots i_p}. \quad (12a)$$

Next let $A_{i_1 \cdots i_p}$ be also of rank $k_2$ on the second index. I shall show that no $B_{ij_1}$ can be of rank greater than $k_2$ on the index $i_2$. To say that $A$ is of rank $k_2$ on $i_2$ is the same as saying that $n - k_2$ linearly independent vectors $X_{q, i_2}$ exist such that

$$A_{i_1, a_{i_2}, \cdots i_p} X_{q, a_2} = 0 \quad (q = 1, 2, \cdots, n - k_2) \quad (13)$$

which by (12a) is the same as

$$\sum_{j_1 = 1}^{j_1 = k_1} a_{ij_1, i_1} B_{j_1, i_1, i_2 \cdots i_p} X_{q, a_2} = 0 \quad (q = 1, 2, \cdots, n - k_2). \quad (14)$$

But since the $k_1$ vectors $a_{ij_1, i_1}$ are linearly independent we must have

$$B_{j_1, a_1 i_2 \cdots i_p} X_{q, a_2} = 0 \quad (q = 1, 2, \cdots, n - k_2) \quad (15)$$

for each value of $j_1$. That is, each $B_{j_1}$ is of rank not greater than $k_2$ on $i_2$ as was to be shown.
A set of $k_2$ vectors $a_{2j}$, where $j_2 = 1, 2, \cdots, k_2$ therefore exists in terms of which every vector $B_{j_1, a_2 i_3 \cdots i_p}$ is linearly expressible. Substituting and collecting terms each $a_{2j}$, for a particular value of $j_1$, will be multiplied into a polyadic of order $p - 2$ which may be called $C_{j_1, j_2}$. Thus each $B_{j_1}$ develops as

$$B_{j_1} = \sum_{j_2=1}^{k_2} a_{2j} C_{j_1, j_2}.$$  \hfill (16)

Then by (12) $A$, if of rank $k_1$ on $i_1$ and rank $k_2$ on $i_2$ can be developed as

$$A = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_p=1}^{k_p} a_{i_1j_1} a_{2j_2} \cdots a_{(p-1)j_{p-1}} g_{j_1 j_2 \cdots j_{p-1}}.$$  \hfill (17)

Proceeding in a similar manner, assuming in succession that $A$ is of rank $k_3$ on $i_3$, etc., up to $k_{p-1}$ on $i_{p-1}$, we arrive at

$$A = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_{p-1}=1}^{k_{p-1}} a_{i_1j_1} a_{2j_2} \cdots a_{(p-1)j_{p-1}} g_{j_1 j_2 \cdots j_{p-1}}.$$  \hfill (18)

where the $g$'s are a set of vectors in the complex of the $p$th index, in number equal to the product $k_1 k_2 \cdots k_{p-1}$. The vectors $a_{mjm}$ are, for a chosen value of $m$, in number $k_m$, that is they lie in a $k_m$-space.

Finally, by similar reasoning, assuming $A$ of rank $k_p$ on $i_p$, the $g$'s lie in a $k_p$-space. The theorem is thus proved.

**Corollary I.** If in (18) each $g$ be expressed in terms of $k_p$ linearly independent vectors $a_{pj}$, we have

$$A = \sum_{j_1=1}^{k_1} \cdots \sum_{j_p=1}^{k_p} a_{i_1j_1} \cdots a_{pj} g_{j_1 \cdots j_p}.$$  \hfill (19)

where the $g_{j_1 \cdots j_p}$ are a set of scalar coefficients in number equal to the product of the ranks $k_1 \cdots k_p$.

**Corollary II.** By a properly chosen transformation $T_{am}$ as in (4) the vectors $a_{mj}$ can be identified with the first $k_m$ basis vectors of the $m$th complex $e_1, e_2, \cdots, e_{k_m}$. Thus without altering
any multiply invariant property, \( A \) can be assumed of the form
\[
A = \sum_{j_1=1}^{j_1=k_1} \cdots \sum_{j_p=1}^{j_p=k_p} e_{j_1} \cdots e_{j_p} \xi_{j_1 \cdots j_p}
\] (20)
provided the ranks on \( i_1 \cdots i_p \) are \( k_1 \cdots k_p \) respectively.

**Corollary III.** If none of the ranks \( k_1 \cdots k_p \) exceeds a single number \( k \), it may be assumed, without altering any multiply invariant property, that \( A \) has the form
\[
A = \sum_{j_1=1}^{j_1=k} \cdots \sum_{j_p=1}^{j_p=k} e_{j_1} \cdots e_{j_p} \xi_{j_1 \cdots j_p}
\] (21)
which is the form of a polyadic in \( k \) dimensions instead of in \( n \) dimensions.

It may be remarked that (20) assumes every component of \( A \) to be zero for which any index \( i_m \) is greater than \( k_m \), and similarly for (21). The problem of finding the minimum \( h \) in (2) is the same as choosing the \( a \)'s in (19) so as to annul as many more components as possible. In other words, a knowledge of the ranks on the indices enables us to reduce the number of terms in (2) from \( n^p \) to \( k_1 k_2 \cdots k_{p-1} \) by an expansion (19) but in general fails to give the minimum \( h \).

In the case of a dyadic, however, the rank on either index is equal to the rank in the usual sense, and (19) merely confirms known results.

Also, that a polyadic can be expressed as a single polyad it is necessary and sufficient that rank on every index be unity.

9. **Multiplex rank.**

Consider once more the \( p \)-way matrix of the \( A_{i_1 \cdots i_p} \). Suppose two chosen indices \( i_r \) and \( i_s \) to be varied, all other indices retaining a chosen set of fixed values. We thus obtain a set of \( n^2 \) components, a *duplex file on* \( i_r i_s \), to be taken in natural order (Art. 7). The \( n^{p-2} \) duplex files on \( i_r i_s \) may be arranged as a two-way matrix having \( n^2 \) rows and \( n^{p-2} \) columns. The columns also are to stand in natural order. Any (two-way) determinant of order \( q \) from this matrix is a *duplex minor* of order \( q \) on the indices \( i_r i_s \) of \( A_{i_1 \cdots i_p} \).

(When confusion might arise minor and rank on a single index
will be distinguished as *simple rank* and *simple minor* on the index in question.)

Similarly, ordering \( n^t \) components selected by varying \( t \) chosen indices yields a *multiplex* (\( t \)-plex) *file* on those indices. The \( n^{p-t} \) multiplex files on a chosen set of indices can be arranged as a two-way matrix of \( n^t \) rows and \( n^{p-t} \) columns, which are to be in natural order. A determinant of order \( q \) from this matrix is a *\( t \)-plex minor of order \( q \)* on the designated indices of \( A_{i_1 \ldots i_p} \). The rank of this matrix is the *\( t \)-plex rank* of \( A \) on the \( t \) chosen indices.

**Lemma.** It is evident that the \( t \)-plex rank of \( A \) on \( t \) chosen indices is equal to the \((p-t)\)-plex rank on the remaining indices.


Let the \( p \) indices of \( A_{i_1 \ldots i_p} \) be divided into \( \pi \) groups of \( t_1, t_2, \ldots, t_{\pi} \) indices each. The selection of indices for each group is arbitrary, i.e., the indices of a group need not be consecutive indices in the series \( i_1 \ldots i_p \). Within each group, however, the indices may stand in the order in which they occur in \( i_1 \ldots i_p \). Any array or summation in which the \( t_m \) indices of a group all run from 1 to \( n \) may be taken in natural order (Art. 7). This is the same as replacing the \( t_m \) indices of a group by a single multipartite index which runs from 1 to \( n^{t_m} \). If \( u_1, u_2, \ldots, u_{\pi} \) are the new indices we have identically

\[
A_{i_1 \ldots i_p} = A_{u_1 \ldots u_{\pi}}. \tag{22}
\]

This rearrangement of indices is equivalent to treating a polyadic as a *multiple polyadic*, i.e., a sum of multiple polyads, wherein factors are polyadics instead of vectors. By mere grouping of terms in (3) we can expand as

\[
A_{i_1 \ldots i_p} = \sum_{j=1}^{h} B_{ij, u_1} B_{jj, u_2} \cdots B_{jj, u_{\pi}} \tag{23}
\]

where, for a chosen value of \( m \), the \( B_{m, u_m} \) are a set of \( h \) polyadics of order \( t_m \), the *polyadic factors on the \( m \)th multipartite index*.

The problem of finding the minimum value of \( h \) in (23) is more general than the problem of expressing \( A \) as the sum of a minimum number of products as defined in Art. 1, because now, on the right of (23), the order of indices \( i_1 \ldots i_p \) is no longer preserved.
We may say, however, that if \( A_{i_1 \cdots i_p} \) is developed as in (23) we have thereby expressed an *isomer* of \( A \) as a sum of products in the sense of Art. 1.  

**Lemma.** For an assigned value of \( h \), the possibility of (23) is a multiply invariant property of \( A \).

For by a transformation (4) on any index, any polyadic factor goes into a polyadic of the same order.

It follows that any relationships on which the possibility of (23) depends have multiple invariant character.

It also follows that if (23) exists for a polyadic, it exists for any tensor whose components at the point in question and with the variables employed are respectively equal to those of the polyadic.

11. Development according to multiplex rank.

The polyadic factors \( B_{mij}, u_m \) of (23) are, with respect to all questions of linear dependence, precisely the same as vectors in a space of \( n^m \) dimensions; for a fixed value of \( m \), any \( n^m+1 \) of them are linearly related. If, in a particular case, it holds true that any \( k_m+1 \) of them are linearly related, these polyadics may be said to lie in a \( k_m \)-space (although this is not free from objection).

**Theorem III.** If the polyadic factors on the \( m \)th multipartite index lie in a \( k_m \) space, the \( t_m \)-plex rank of \( A_{i_1 \cdots i_p} \) on the indices of the \( m \)th group is not greater than \( k_m \).

Proof. The \( h \) polyadics \( B_{mij}, u_m \) are linearly expressible in terms of \( k_m \) properly chosen polyadics bearing the index \( u_m \). Let these be \( C_r, u_m \) where \( r = 1, 2, \cdots, k_m \). Substituting for the \( B_{mij}, u_m \) their values and collecting terms, the coefficient of each \( C_r, u_m \) will be a polyadic of order \( p-t_m \) which may be called

\[
D_{r, u_1 \cdots u_{m-1} u_m +1 \cdots u_r}
\]

so that we may write

\[
A_{i_1 \cdots i_p} = \sum_{r=1}^{r=k_m} C_{r, u_m} D_{r, u_1 \cdots u_{m-1} u_m +1 \cdots u_r}
\]

(24)

\( ^4 \) An isomer (conjugate) is a new polyadic or tensor obtained from \( A_{i_1 \cdots i_p} \) by a permutation of indices.
Therefore by the laws of matrix multiplication $A$ is of rank not greater than $k_m$ on the multipartite index $u_m$ as was to be shown.

**Theorem IV.** With indices in $\pi$ groups, if $A_{i_1 \ldots i_p}$ is of multiplex rank $k_1$ on the indices of the first group, $k_2$ on those of the second group, and so on up to $k_\pi$ on those of the $\pi^{th}$ group, the polyadic or tensor can be developed in the multiple polyadic form (23) in such a manner that, simultaneously, the polyadic factors of the $m^{th}$ multipartite index lie in the $k_m$-space for $m=1, 2, \ldots, \pi$.

Proof. Let $\beta_1, \beta_2, \ldots, \beta_\pi$ be dummy indices corresponding to the multipartite indices $u_1, u_2, \ldots, u_\pi$. Furthermore let the (ordinary) indices of the first group have the dummies $\rho_1, \ldots, \rho_{t_1}$. Let the polyads $e_{\rho_1} e_{\rho_2} \cdots e_{\rho_{t_1}}$ in natural order, be $E_{\beta_1}$, so that as $\beta_1$ runs from 1 to $n^{t_1}$ these polyads all occur once and once only. Treating the other groups similarly, an isomer $A'$ of $A$ can be written

$$A' = A_{\beta_1} \cdots A_{\beta_\pi} E_{\beta_1} \cdots E_{\beta_\pi}$$

identically. Thus $A'$ takes the form of a polyadic of order $\pi$. The proof now proceeds as in Theorem II, multiplex rank replacing simple rank, and multipartite indices behaving like ordinary indices of that proof. We arrive at a development of $A'$ which has precisely the form of the right member of (18) but wherein the factors, instead of being vectors, are polyadics of orders $t_1, t_2, \ldots, t_\pi$. Restoring the original order of indices we have a development of the form (23) in which any set of polyadic factors $B_{mj}, u_m$ lies in a $k_m$-space. The theorem is therefore proved.

**Corollary I.** With the hypotheses of Theorem IV we may write

$$A_{i_1 \ldots i_p} = A_{u_1 \ldots u_\pi} = \sum_{j_1=1}^{k_1} \cdots \sum_{j_\pi=1}^{k_\pi} B_{ij_1}, u_1 \cdots B_{\pi j_\pi}, u_\pi g_{j_1 \cdots j_\pi}$$

where the $B_{mj}, u_m$ are polyadics or tensors of order $t_m$ and the $g_{j_1 \cdots j_\pi}$ are a set of scalar coefficients in number equal to the product of the multiplex ranks $k_1 \cdots k_\pi$.

It may be remarked that we have here no corollary analogous to Corollary II of Theorem II. We cannot in general by linear transformations (4) identify a set of polyadic factors $B_{mj}, u_m$ with the first $k_m$ polyads $E_1 \cdots E_{km}$. 
Somewhat analogous with Theorem II, Corollary III we have

**Corollary II.** If none of the multiplex ranks $k_1 \ldots k_\pi$ exceeds a single number $k$, a suitably chosen isomer $A'$ of $A$ can be written

$$
A' = \sum_{j_1=k} \ldots \sum_{j_\pi=k} B_{ij_1} B_{ej_2} \ldots B_{\pi j_\pi} g_{j_1 \ldots j_\pi} \tag{27}
$$

which is the form of a polyadic in $k$ dimensions instead of $n$ dimensions. Any set of $k$ polyadic factors $B_{m_jn}$ can be so chosen as to be linearly independent, hence are like basis vectors in a $k$-space (this of course in a purely algebraic sense). The $g$'s are scalar coefficients, analogous to the components of a tensor in the $k$-space. The utility of this analogy between polyadic factors and vectors will appear shortly.

**Corollary III.** The simple or multiplex rank of a polyadic on one or several indices is a multiple invariant.

For by Theorem IV the possibility of a particular development of form (23) depends on such rank, and this form is multiply invariant.

When $\pi=2$ we have a case of (23) treated by Dr. J. W. Alexander.\(^5\)

When $\pi=2$ and $p$ is even, with $t_1=t_2=k$, the indices of the first group being $i_1 \ldots i_k$, in order, we have "double polyadics".\(^6\) In this case $g_{j_1,j_2}$ behaves like a dyadic and (23) becomes analogous to (5).

Again, that a polyadic or one of its isomers may be reduced to a product of polyadics of lower order it is necessary and sufficient that the rank on each group of indices shall be unity for at least one manner of grouping.

12. **Character of the problem when $\pi$ exceeds 2.**

When $\pi$ exceeds 2 a knowledge of the ranks on various groups of indices, while quite essential, does not in general solve the problem of the minimum value of $h$ in (23). Reduction to fewer

\(^5\) Loc. cit. note 2.

\(^6\) Proc. Amer. Acad. Arts and Sci., Vol.58, No. 10, May 1923. The multiple dot products of this paper may be regarded as mere summations on multipartite indices $u_i$ and $u_\beta$. 
than $k_1 \ldots k_r$ terms depends on invariants of another character, the quantities elsewhere denoted by $\mathbf{P}_k ((\mathbf{P}^n))$ where $\mathbf{P}$ is the polyadic. They are always expressible as $p$-way determinants, and, from their very form, are multiply invariant. These statements I shall illustrate in the next article by a study of the simplest possible case where $\pi$ exceeds 2, namely the problem of expressing a triadic in two-space as a sum of two triads.

13. The case $\pi = p = 3$, $n = 2$, $h = 2$.

We have to solve the polyadic equation

$$A_{a_1 a_2 a_3} e_{a_1} e_{a_2} e_{a_3} = a_1 a_2 a_3 + b_1 b_2 b_3$$

where the components $A_{a_1 a_2 a_3}$ are 8 given numbers and the 6 vectors on the right are required. This is equivalent to 8 ordinary equations

$$A_{i_1 i_2 i_3} = a_{i_1} a_{i_2} a_{i_3} + b_{i_1} b_{i_2} b_{i_3}$$

where the 12 numbers on the right are unknown.

To solve, equate values of the covariant $F_1 ((\mathbf{A}^3))$ obtained from the two members of (28). The operation $F_1$ implies two steps: first, multiply each member into itself, preserving the natural order of indices (multiple indeterminate product); second, replace each group of vectors of like index by its space complement (folding), excepting the third index.

Thus from the right of (28) we first form the product

$$a_1 a_2 a_3 a_4 + a_1 b_2 b_3 a_4 + b_1 a_2 a_3 b_4 + b_1 b_2 b_3 a_4$$

We then take the space complements of the 8 dyads $a_1 a_2$, $a_1 b_2$, $b_1 a_2$, $b_1 b_2$, which, because $n = 2$, are the respective determinants of the components of the vectors concerned. The space complements of the first, second, seventh, and eighth of these dyads therefore vanish and the result is

$$(a_1 b_1)(a_2 b_2)a_3 b_3 + (b_1 a_1)(b_2 a_2)b_3 a_3$$

where $(a_1 b_1)$, etc., are determinants. This may be written

$$(a_1 b_1)(a_2 b_2)[a_3 b_3 + b_3 a_3]$$

where the quantity in brackets is a symmetrical dyadic.

7Loc. cit. note 1, p. 233.
The same process applied to the left member of (28) yields a cubic determinant with vector elements which develops\(^8\) into the symmetrical dyadic

\[ e_1 e_1 m_{11} + (e_1 e_2 + e_2 e_1) m_{12} + e_2 e_2 m_{22} \]  

(32)

where

\[ m_{11} = 2(A_{111}A_{222} - A_{121}A_{211}), \quad m_{22} = 2(A_{112}A_{222} - A_{122}A_{212}) \]
\[ m_{12} = A_{111}A_{222} - A_{211}A_{122} - A_{121}A_{212} + A_{221}A_{112} \]  

(33)

so that by equating the two values of \( F_1((A^2)) \) from (31) and (32)

\[ e_1 e_1 m_{11} + (e_1 e_2 + e_2 e_1) m_{12} + e_2 e_2 m_{22} = (a_1 b_1)(a_2 b_2) [a_3 b_3 + b_3 a_3] \]  

(34)

We now consider three cases.

**Case I.** The binary quadric

\[ x_1^2 m_{11} + 2x_1 x_2 m_{12} + x_2^2 m_{22} \]  

(35)

does not vanish identically and has linearly independent factors.

Let these factors be

\[ x_1 r_3 - x_2 r_3' \quad \text{and} \quad x_1 s_3 - x_2 s_3' \]  

(36)

and assume

\[ a_3 = e_1 r_3 - e_2 r_3' \quad \text{and} \quad b_3 = e_1 s_3 - e_2 s_3' \]  

(37)

These two vectors not being parallel we may certainly write

\[ A = X a_3 + Y b_3 \]  

(38)

where \( X \) and \( Y \) are dyadics. I shall now show that \( X \) is the product of two vectors, hence may be written \( a_1 a_2 \), and similarly for \( Y \); so that (38) takes the required form (28).

**Proof.** Let the result of multiplying \( A \) into \( b_3 \) and taking space complements of \( b_3 \) with the postfactors of \( A \) be denoted\(^9\) by \( A \times b_3 \). Thus from (38)

\[ A \times b_3 = X(a_3 b_3) \]  

(39)

both members of which are dyadics. The operation \( A \times b_3 \) is the same as replacing \( e_1 \) and \( e_2 \), basis vectors of the third index in \( A \), by \(-s_3'\) and \(-s_3\) respectively; for \((e_1 b_3) = -s_3'\) and \((e_2 b_3) = -s_3\), by (37). Multiply each side of (39) into itself and fold on both factors, i.e., perform the operation \( F_0 \). The right side gives

\(^8\) Loc. cit. note 1, p. 235.  

\(^9\) Using the notation of Dr. Naess, which is consistent because the space complement is a generalization of Gibbs' "cross product."
\[(a_3b_3)^2 F_0((X^2)), \text{ that is } (a_3b_3)^2 X \otimes X,\text{ which is } 2(a_3b_3)^2 |X| \text{ where } |X| \text{ is the determinant of } X.\] The left side gives the same result as making the above mentioned replacement in (32), i.e., \(-s_3'\) for \(e_1\) and \(-s_3\) for \(e_2\), because it is immaterial whether we make this replacement before or after folding \(A\) on its first two indices; but this result is zero, by definition of \(b_3\) in (37). Hence \(|X| = 0\) and \(X\) can be written as a dyad. Similarly \(|Y| = 0\) and the required form (28) is established.

It has thus been shown that (28), or (28a), is a possible development for \(A_{i_1i_2i_3}\) (with \(n = 2\)) provided (35) has linearly independent factors; i.e., if we write
\[I_a = 2m_{12}m_{21} - m_{13}m_{31}, \quad (40)\]
this sufficient condition is that \(I_a\) be not zero. If we denote (32) by \(M\) we have
\[F_0((M^2)) = I_a, \quad (41)\]
that is \(I_a = M \otimes M\), a multiple invariant of \(A\). Comparing with (34) we have
\[I_a = 2(a_3b_3)^2(a_2b_2)^2(a_1b_1)^2, \quad (42)\]

**Case II.** \(I_a = 0\) and \(A\) is of rank unity on at least one index.

The form (28) is at once established by Theorem IV. For suppose \(A\) of rank 1 on \(i_3\). We may write \(A = Ba\) where \(B\) is a dyadic and \(a\) is a vector. \(B\) may certainly be written as the sum of two dyads, hence (28) exists, with \(a_3 = b_3 = a\). Similarly (28) exists if \(A\) is of rank 1 on \(i_1\) or on \(i_2\).

**Case III.** \(I_a = 0\) and \(A\) is of rank 2 on all indices.

The form (28) is now impossible, for if (28) exists (42) also exists. If \(I_a = 0\) one of the three determinants on the right must vanish. But then, by (28) \(A\) must be of rank 1 on some index, contrary to hypothesis.

For example, if the \(A_{i_1i_2i_3}\), taken in natural order (Art. 7) are the first 8 integers 1, 2, \(\cdots\), 8, it will be found that \(I_a = 0\) while ranks on \(i_1\), \(i_2\), and \(i_3\) are all 2, and (28) is impossible.

**Remarks on the above cases.** It is evident that every triadic in two dimensions which does not vanish identically must belong to one of the above three types, the classification being with reference to rank on the indices and to \(I_a\), all of which are multiple
invariants. Thus the classification applies without change to any tensors of the third order when \( n = 2 \). This relatively simple example illustrates very well the statement of Art. 12 that classification of tensors and polyadics (with respect at least to their expression as sums of products) depends on quantities \( F_k((P^2)) \).

14. **Case \( \pi = \rho = 3, h = 2, n \) arbitrary. Reduction of any triadic or third order tensor to a sum of two triads.**

The results of Art. 13 may be at once extended to any triadic or third order tensor by virtue of the following:

**Theorem V.** If the polyadic development (2) exists for any assigned value of \( h \), the simple rank of \( A \) on any index cannot exceed \( h \).

For there are only \( h \) vector factors of any one index. Hence these factors lie in an \( h \)-space, and, by Theorem I, the rank on this index is not greater than \( h \).

If, therefore, we assume \( h = 2 \) for any triadic, the rank on any index can be at most 2. By Corollary III of Theorem II we may reduce the triadic to the form of a triadic in 2-space without altering any multiply invariant property. The entire argument of Art. 13 may then be applied.

What \( I_a \) now becomes deserves mention. This quantity was formed by first taking \( M = F_1((A^2)) \), which may be indicated by the briefer notation \( (A^2)_1 \). Then \( I_a = F_0((M^2)) = M \otimes M \). Thus we may briefly write

\[
I_a = (A^2)_1 \otimes (A^2)_1. \tag{43}
\]

The operations \( F_0 \) and \( F_1 \) are both multiply invariant. Applying the results of Art. 13 we have the result: any triadic or third order tensor whose rank on any index does not exceed 2 may be expressed as the sum of two triads except when the quantity \( (A^2)_1 \otimes (A^2)_1 \) vanishes and none of these ranks is unity. It should be noticed that this quantity is no longer a scalar.

15. **Case \( \pi = 3, h = 2, n \) and \( \rho \) arbitrary. Reduction of any polyadic or tensor to a sum of two terms each of which is the product of three polyadics or tensors of lower order.**

In a similar manner, the argument of Art. 13 applies to any polyadic or tensor when we divide the indices into three groups and assign \( h = 2 \) in the development (23). This is by:
Theorem VI. If the multiple polyadic development (23) exists for an assigned value of \( h \), the multiplex rank of \( A \) on any multipartite index cannot exceed \( h \).

For there are only \( h \) polyadic factors of any one multipartite index. Hence by Theorem III the multiplex rank on this index is not greater than \( h \).

If therefore, with three multipartite indices, we assume \( h = 2 \) in (23), the multiplex rank on none of these indices exceeds 2. By Corollary II of Theorem IV we may reduce \( A \) (or an isomer) to the form of a triadic in 2-space. The argument of Art. 13 may then be applied, the scalars \( g_{j_1j_2j_3} \) (cf. (27)) replace the \( A_{i_1i_2i_3} \) of Art. 13 and the \( B \)'s replace basis vectors.

Interesting questions arise when we consider a generalization of \( I_a \) for this case, to be obtained from the original \( A \) instead of using the \( g \)'s. The form would be that of the right of (43), but the folding operations apply to polyadic factors instead of to vectors. It can be shown that the results of these operations are still invariants. Applying Art. 13 we have the result: any polyadic or tensor may be expressed as the sum of two products of the form (23) if indices may be taken in three groups in such a manner that no multiplex rank on these groups exceeds 2, with the following exceptional case: when none of these ranks is unity and the quantity \( (A^2)_{1}^{\infty}(A^2)_{1} \), in which the folding operations refer to the multipartite indices, vanishes.

16. Case \( n = h = 2 \). Reduction of a tetradic or fourth order tensor in 2-space to the sum of two tetrads.

We now consider the next simplest possibility, a tetradic in two-space \( n = 2 \) and desire to express it as the sum of two tetrads,

\[
A = a_1a_2a_3a_4 + b_1b_2b_3b_4
\]  

(44)

or

\[
A_{i_1i_2i_3i_4} = a_{i_1i_2i_3i_4} + b_{i_1i_2i_3i_4}
\]  

(44a)

a system of 16 equations, the 16 numbers on the right unknown. As a first step in the solution, note the following:

Theorem VII. If the polyadic development (2) exists for an
assigned value of \( h \), the multiplex rank of \( A \) on any group of indices cannot exceed \( h \).

For grouping the vector factors into polyads to correspond with the grouping of indices, (2) takes the form (23) with the same value of \( h \). There are now \( h \) polyad factors on any one multipartite index. Hence by Theorem III the multiplex rank on this index is not greater than \( h \).

In the present case it follows that, if (44) exists, the respective duplex ranks on \( i_1i_2 \), \( i_1i_3 \), and \( i_1i_4 \) do not exceed 2.

Next, equate values of \( \Gamma_0((A^2)) \) from the two members of (44), i.e., fold each side into itself on both factors, giving

\[
|A_{i_1i_2i_3i_4}| = (a_1b_1)(a_2b_2)(a_3b_3)(a_4b_4) \quad (45)
\]

where \( |A_{i_1i_2i_3i_4}| \) is the four-way full-sign determinant\(^{11}\) of the components of \( A \), that is

\[
|A_{i_1i_2i_3i_4}| = A_{1111}A_{2222} - A_{1112}A_{2221} - A_{1121}A_{2212} + A_{1122}A_{2211} - A_{1211}A_{2122} + A_{1212}A_{2121} + A_{1221}A_{2112} - A_{1222}A_{2111}. \quad (46)
\]

We now consider three cases.

**Case I.** The determinant (46) is not zero.

Since the duplex rank on \( i_1i_2 \) must not exceed 2 we may write by Theorem IV

\[
A = B_1C_1 + B_2C_2 \quad (47)
\]

where the factors on the right are dyadics. I shall now show that \( B_1 \) and \( B_2 \) may be so chosen as to be dyads, and that when they are so chosen \( C_1 \) and \( C_2 \) are also dyads, thus establishing (44).

**Proof.** At least one of the dyadics \( B_1, B_2 \) can be taken as a dyad. For if \( B_1 \) is not a dyad at least one dyadic of the pencil \( B_1 + xB_2 \) is a dyad, since equating the determinant of the pencil to zero gives a quadratic in \( x \) having at least one root. Take this dyad instead of \( B_1 \). Thus assuming \( B_1 \) a dyad, by a suitable transformation (4) on \( i_1 \) and another on \( i_2 \) we can identify \( B_1 \) with \( e_1e_1 \); and (46), being a multiple invariant,\(^{12}\) will not be made to vanish. Assume \( B_2 = b_{12}e_1e_2 + b_{22}e_2e_1 + b_{22}e_2e_2 \). If \( b_{22} \) is not zero a dyadic

\(^{11}\) Loc. cit. note 1, p. 234, Theorem IX.

\(^{12}\) Loc. cit. note 3, p. 246, Ex. 1.
can be chosen from the pencil $B_1 + xB_2$ which shall be a dyad. Take this instead of $B_2$.

If $b_{22}$ could be zero, by transformations (4) we could make $B_2 = e_1e_2 + e_2e_1$ and then write by (47)

$$A = e_1e_1(A_{111} + e_1e_2A_{112} + e_2e_1A_{121} + e_2e_2A_{122})$$

$$+ (e_1e_2 + e_2e_1)(e_1e_1A_{211} + e_1e_2A_{212} + e_2e_1A_{221} + e_2e_2A_{222}).$$ (48)

The two-way matrix of the duplex files on $i_1i_3$ is now

$$\begin{bmatrix}
A_{111} & A_{112} & A_{121} & A_{122} \\
A_{121} & A_{122} & A_{211} & A_{212} \\
A_{211} & A_{212} & 0 & 0 \\
A_{221} & A_{222} & 0 & 0
\end{bmatrix}$$ (49)

If this is of rank not exceeding 2 we must have

$$A_{121}A_{122} = 0.$$

But this reduces $|A_{i_1i_3i_4}|$ to zero for this determinant is now $-2(A_{121}A_{122} - A_{121}A_{122})$. Hence (48) is impossible and our assumption that $b_{22} = 0$ is contradictory. Thus $b_{22}$ is not zero and $B_2$ can be taken as a dyad.

Taking $B_1$ and $B_2$ as dyads, by suitable transformations (4) we can identify these with $e_1e_2$ and $e_2e_1$. By (47) we may write

$$A = e_1e_2(e_1e_1A_{121} + e_1e_2A_{122} + e_2e_1A_{211} + e_2e_2A_{212})$$

$$+ e_2e_1(e_1e_2A_{121} + e_1e_2A_{122} + e_2e_1A_{211} + e_2e_2A_{212}).$$ (50)

The two-way matrix of duplex files on $i_1i_3$ is now

$$\begin{bmatrix}
0 & 0 & A_{121} & A_{122} \\
0 & 0 & A_{211} & A_{212} \\
A_{211} & A_{212} & 0 & 0 \\
A_{221} & A_{222} & 0 & 0
\end{bmatrix}$$ (51)

whence by inspection the determinants of $C_1$ and $C_2$ vanish if this is of rank 2. $C_1$ and $C_2$ may be taken as dyads, and (47) takes the form (44). Thus it has been shown that (44) exists if $|A_{i_1i_3i_4}|$ is not zero and the duplex ranks are 2. (It may be noted that if none of these ranks exceeds 2 while some one of them is unity, the determinant (46) vanishes, as is easily proved.)
Case II. The four-way determinant (46) vanishes and one of the simple ranks is unity.

By Theorem IV we may now write

\[ A_{i_1i_2i_3i_4} = a_r B_{stu} \] (52)

where \( a_r \) is a vector whose index is that one of the four indices \( i_1i_2i_3i_4 \) on which the rank is unity, and \( B_{stu} \) is a triadic whose indices correspond to the remaining three. The solution may then be completed by Art. 13.

Case III. The determinant (46) vanishes while all simple ranks are 2. By (45) the required form (44) is then impossible.

Remarks on the above cases. The solution of Case I was arrived at without assuming the duplex rank on \( i_1i_4 \) to be 2, but this is a necessary condition. Hence it follows that if neither of the ranks of \( i_1i_2 \) nor \( i_1i_3 \) exceeds 2, the same is true of that on \( i_1i_4 \). It also follows from the method of proof that, with these ranks not exceeding 2, the condition that the pencil \( xB_1 + yB_2 \), in the notation of (47), shall contain only one dyad is the vanishing of (46).

The four-way determinant (46) in this example plays the same part as did \( I_a \) in Art. 13, an interesting analogy. The difference in form of these two invariants illustrates the contrast between polyadics of odd and of even order.

17. The case \( \pi = 4, h = 2, n \) and \( p \) arbitrary.

As in articles 14 and 15 we may at once generalize to any tetradic and further to any polyadic with indices in four groups. The necessary conditions being imposed upon the ranks, and development made by (21) or (27), the reasoning of Art. 16 applies without formal change.

18. Case \( \pi = p = n = h = 3 \). Reduction of a triadic or third order tensor in three-space to a sum of three triads.

Taking \( n = 3 \) the simplest case which is of interest arises when we assign \( h = 3 \). For, as shown in Art. 14, if we assign \( h = 2 \) the necessary conditions on the ranks practically reduce the problem to a binary one. Let

\[ A = a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 \] (53)
equivalent to 27 equations. As in Art. 13 equate values of $F_1((A^2))$ from both sides. The right side gives

$$(a_1b_1)(a_2b_2)[a_3b_3 + b_3a_3] + (b_1c_1)(b_2c_2)[b_3c_3 + c_3b_3] + (c_1a_1)(c_2a_2)[c_3a_3 + a_3c_3]$$

where the space complements $(a_1b_1)$, etc., are now vectors because $n=3$, and may be denoted by $a_1 \times b_1$, etc., whenever convenient. The quantities in brackets are symmetrical dyadics. Hence (54) is a tetradic. The left of (53) gives a cubic determinant\(^\text{13}\) which develops as

$$e_a e_\beta C_{a\beta}$$

where the $C_{a\beta}$ are a set of 9 symmetrical dyadics; their scalar components resemble $m_{11}$, $m_{12}$, and $m_{13}$ of Art. 13. We may write

$$C_{ab} = e_r e_s m_{abr}$$

where

$$m_{11rs} = A_{22r}A_{33s} - A_{23r}A_{32s} + A_{32r}A_{23s} + A_{33r}A_{22s}$$

$$m_{12rs} = -A_{21r}A_{33s} + A_{23r}A_{31s} + A_{31r}A_{23s} - A_{33r}A_{21s}$$

and so on. Thus there are 54 of these $m$'s, cubic determinants of the second order, becoming twice a flat determinant when $r=s$. And $m_{abr} = m_{abr}$.

If the basis vectors of the third index $i_3$ of $A_{i_1i_2i_3}$ be replaced by a set of scalar variables $x_{i_3}$, the vectors $a_3$, $b_3$, and $c_3$ become linear forms $a_3 x_{i_3}$, etc., which may be abbreviated $a_x$, $b_x$, and $c_x$. The vectors $(a_1b_1)$, etc., may be abbreviated $r_1$, $p_1$, $q_1$ and $r_2$, $p_2$, $q_2$. Then (54) becomes $2p_1 p_2 x a x + 2q_1 q_2 a b x + 2r_1 r_2 b c x$. The 9 symmetrical dyadics $C_{a\beta}$ becomes 9 ternary quadrics which may be written $C_{a\beta}$ and regarded as conics. Equating results from (54) and (55) we then have

$$e_a e_\beta C_{a\beta} = 2p_1 p_2 c x a x + 2q_1 q_2 a b x + 2r_1 r_2 b c x$$

equivalent to 9 equations which must hold if (53) exists. Evidently any four of the nine conics must be linearly related, and, when three conics $C_1$, $C_2$, and $C_3$ are chosen in terms of which all\(^\text{13}\)Loc. cit. note 1, p. 227, determinant (28) with the two triadics equal. The method of development is very fully exemplified in that paper.
may be expressed, their Jacobian must have three linear factors or else vanish identically. These are necessary conditions which the $A_{i_1i_2i_3}$ must satisfy if (53) is possible.

Conversely, assume the nine conics expressible in terms of three conics $C_1, C_2, C_3$, whose Jacobian has three factors not linearly related, $a_x, b_x,$ and $c_x$. Let $a_3, b_3, c_3$ be the vectors obtained by restoring $e_1, e_2,$ and $e_3$ in place of $x_1, x_2, x_3$. We may write

$$A = Xa_3 + Yb_3 + Zc_3$$  \hspace{1cm} (60)

where $X, Y,$ and $Z$ are dyadics. I shall show that $X$ is the product of two vectors, hence may be written $a_1a_2,$ and similarly for $Y$ and $Z$; so that (60) takes the required form (53).

Proof. Let the result of multiplying $A$ into $b_3c_3$ and taking space complements with the postfactors of $A$ be denoted by $A^1 \times b_3c_3$. Thus by (60)

$$A^1 \times b_3c_3 = X(a_3b_3c_3)$$  \hspace{1cm} (61)

both members of which are dyadics. The operation $A^1 \times b_3c_3$ is the same as replacing $e_1, e_2,$ and $e_3$, basis vectors of the third index in $A$, by their cofactors in the determinant

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ b_{3,1} & b_{3,2} & b_{3,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{vmatrix}.$$  \hspace{1cm} (62)

Multiply each side of (61) into itself and fold on both factors, i.e., perform the operation $F_0$. The right side gives $(a_3b_3c_3)^2F_0((X)^2$, that is $(a_3b_3c_3)^2X^X$, whose vanishing is necessary and sufficient that $X$ be the product of two vectors. The left side gives the same result as making the above replacement in (56), because it is immaterial whether we make it before or after folding $A$ on its first two indices; but this result is zero, by definition of $b_3$ and $c_3$, (three conics pass through double points of their Jacobian). Hence $X$ can be written as a dyad, and similarly for $Y$ and $Z$; and the required form (53) is established.

Remarks on the necessary conditions for (53). To say that any four of the nine conics $C_{a3}$ are linearly related is the same as saying that the four-way matrix of the components of the tetradic $F_1(A^2)$ is of duplex rank 3 on its first two indices, the 81 (54 dis-

\textsuperscript{14}Again using the notation of Dr. Naess.
tinct) quantities \( m_{abr} \) being arranged in a two-way matrix as described in Art. 9. The \( m_{abr} \) are cubic determinants picked from the original matrix \( A_{ij,ik} \), cubic minors, we may say, of the second order.

The Jacobian of any three of the nine conics must vanish identically or have three linear factors. It is well known that a cubic has three linear factors when and only when it is proportional to its own Hessian. Hessians and Jacobians thus arising out of the \( m_{abr} \) are components of polyadics obtainable by further folding of \( F_1(A^2) \). Special cases arise according to whether the linear factors are linearly dependent, or, perhaps, all Jacobians vanish identically.

19. Conclusion. Continuance of the problem requires further extension of the notion of rank, which would too greatly lengthen the present paper. The simple and duplex ranks which have so far arisen have been two-way, in the sense that the \( p \)-way matrix of the \( A_{i_1..i_p} \) was arranged in some manner as a flat matrix in defining these ranks. But to continue the present problem it is necessary to consider the vanishing of \( k \)-way determinants picked from the \( p \)-way matrix without flattening it.

A number of relations exist among the simple or multiplex ranks already defined, some of which were suggested in the course of the work. These appear more clearly if the matrix is not flattened, i.e., a multiplex file, instead of being straightened out and treated like a vector, is left in position, as a couche in the original matrix. In this paper the analogy of a multiplex file with a vector has been expressly emphasized, bringing out the analogy of Theorem IV with Theorem II; this is undoubtedly somewhat artificial, but it enables us to extend results when \( \pi = p \) immediately to arbitrary \( p \), with the same value of \( \pi \), by virtue of (27).

Finally, a word as to the space complement and notations for expressing it appears necessary. Any space complement, by Dr. Naess' definition, is a determinant. Its components can be expressed by summation with the \( \epsilon_{r_1..r_n} \), etc., of Ricci and Levi-Civita. The

15 My colleague, Mr. L. H. Rice, to whom I am indebted for many helpful suggestions, will treat these relations in a paper shortly to appear.
determinant of Naess having elements of every row the basis vectors \( e_1 \cdots e_n \) is analogous to the pseudo scalar of Grassmanian analysis. What notation we employ is partly a matter of convenience, but should embody, if possible, those analogies which are of most importance in the problem concerned and in the logic employed. In the algebra of tensors or polyadics, of which the present problem is a part, the most important analogy, in the author's opinion, is with the symbolic method of handling invariants and other concomitants of quantics. Real vector factors behave like the symbolic factors of that method. Folding is analogous with "Faltung," due regard being had for the non-commutative character of the symbols. The quantities \( F_k((A^0)) \) are analogous with transvectants. This is why the notation \( (a_1a_2a_3) \), etc., has been employed for space complements.