Geometry of the eigenvectors of plane rotation

Rotation of a two-dimensional vector (x,y) in the Cartesian plane can be expressed as a matrix multiplication, where the rotated vector (x',y') is equal to the product of the rotation matrix M and the original (column) vector (x,y). For rotation by a counter-clockwise angle A, the antisymmetric 2x2 matrix M has the cosine of A in the diagonal elements, the sine of A at lower left, and the negative of the sine of A at upper right. It is often useful to find the eigenvalues and eigenvectors of a matrix transformation (i.e., the constant and vector with the property that multiplication of the vector by the matrix or by the constant gives the same result) and this can be done for M. A standard calculation -- setting the determinant of the difference matrix M - cI [where I is the identity matrix] equal to zero and solving for the eigenvalue c -- shows that the eigenvalues are c = exp(iA) and c' = exp(-iA)=c*, where i is the square root of -1 and * denotes complex conjugate. Thus, M has real eigenvalues only when A is an integer multiple of pi, in which case c = 1 or c = -1: only when the rotation angle is 0 or 180 degrees does rotation in the plane return the original vector scaled by a constant; and in either of these special cases, any vector is an eigenvector.

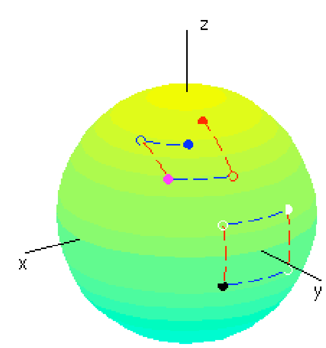
For an arbitrary angle A, however, M has the two complex-conjugate eigenvalues c = exp(iA) and c* = exp(-iA). These complex eigenvalues have meaning only if the setting is generalized to allow x and y to be complex, so that the products of c and x and of c and y are again elements of the vector space. The product of c and a complex number z is itself equivalent to a rotation of z in the complex plane by the angle A: to see this, represent z in polar coordinates as magnitude times exponential of the product of i and the angle; then multiplying by c just adds A to the angle. Thus, multiplication of the complex eigenvector (x,y) by the eigenvalue c or c* is equivalent to rotation of the complex x and y by the angle A (for c) or the angle -A (for c*) in the respective complex x and y planes.

The matrix M, however, is real. Thus, for complex x and y, the product of M and x or y can be viewed as the product of M and the real vector Re(x,y) plus i times the product of M and the real vector Im(x,y), where Re and Im denote the real and imaginary parts, respectively. Each of these real products is a rotation by A in a real two-dimensional plane: the first is the rotation by A in the plane of the real parts of x and y, and the second is the rotation by A in the plane of the imaginary parts of x and y. Each of these is equivalent in form to the original rotation of the original real vector (x,y).

The eigenvalue condition in the complex setting can therefore be viewed as a commutation relation: rotation by A of the vector of real parts of x and y in the plane Im(x,y)=(0,0), followed by rotation by A of the vector of imaginary parts of x and y in the plane Re(x,y)=0, gives the same result as rotation of the complex x by A (or -A, for c*) in the plane y=0, followed by rotation of the complex y by A (or -A, for c*) in the plane x=0. The eigenvectors corresponding to c and c* are those vectors that satisfy this commutation relation. This is different from the real eigenvalue case, in which the matrix multiplication returns the "same" eigenvector, multiplied only by a scalar; this real eigenvalue problem could also be viewed as a commutation relation, but the scalar multiplication is nearly trivial.

The vector space in the complex setting is effectively four dimensional, with independent real and imaginary parts for each of x and y. This makes the geometry of the rotations by M and c or c* difficult to depict or visualize. A useful analog is the transformation induced by two successive rotations about orthogonal axes in three-dimensional space, with coordinates (x,y,z). Consider a rotation by angle A1 about the z axis, followed by rotation by angle A2 about the x axis. Alternatively, rotate first by A2 about the x axis, and then by A1 about the z axis. The matrix M1 for the rotation by A1 has the matrix M in the upper-left 2x2 box (with A=A1), a 1 in the lower-right diagonal element, and zeros elsewhere. The matrix M2 for the rotation by A2 is similar, with M (for A=A2) now in the lower-right 2x2 box and a 1 in the upper-left diagonal element. It is straightforward to find the eigenvectors of this commutation relation -- those (x,y,z) for which the rotations (matrix multiplications) give the same answer for either order -- by setting the difference of the two matrix products equal to zero and computing the non-trivial solutions. An example is shown below (Figure 1). For two points (magenta and black dots), two sets of rotations are shown: first about the z axis (blue dashed lines, to open circles) and then about the x axis (red dashed lines), or first about the x axis (red dashed) and then about the x axis (blue dashed). The two pairs of rotations have different results (blue and red dots) for the magenta point, so that point is not an eigenvector. For the black point, the two pairs of rotations have the same result (white dot), so the black point is an (the) eigenvector.

Figure 1



The commutation relation for the complex, two-dimensional rotation can be illustrated in a similar way. Two such depictions for the eigenvalue c, corresponding to rotation of the complex x and y by the original angle A, are shown below (Figure 2). Each side shows three coordinates: (Re x, Re y, Im x) on the left, (Re x, Re y, Im y) on the right. The rotations by A in the real planes Re(x,y) and Im(x,y) are shown by blue and red dashed lines, respectively. The rotations by A in the complex planes x and y are shown by red and blue dotted lines, respectively. The two pairs of rotations of the magenta point, which is not an eigenvector, give different results (red and blue dots). The two pairs of rotations of the black point, which is an eigenvector, give the same result (white dot on left; green dot on right). The eigenvector in this case is (x,y)=(1,-i)/sqrt(2) [where sqrt(2) denotes the square root of 2], but any vector obtained from this vector by multiplication by a complex number will also be an eigenvector. A similar pair of panels is shown (Figure 3) for the complex-conjugate eigenvalue c* and the complex-conjugate eigenvector (x,y)=(1,i)/sqrt(2). Note that with the usual complex-conjugate inner-product (so that the inner product of two complex vectors is the first times the complex-conjugate of the second), these two eigenvectors are orthogonal.

Figure 2

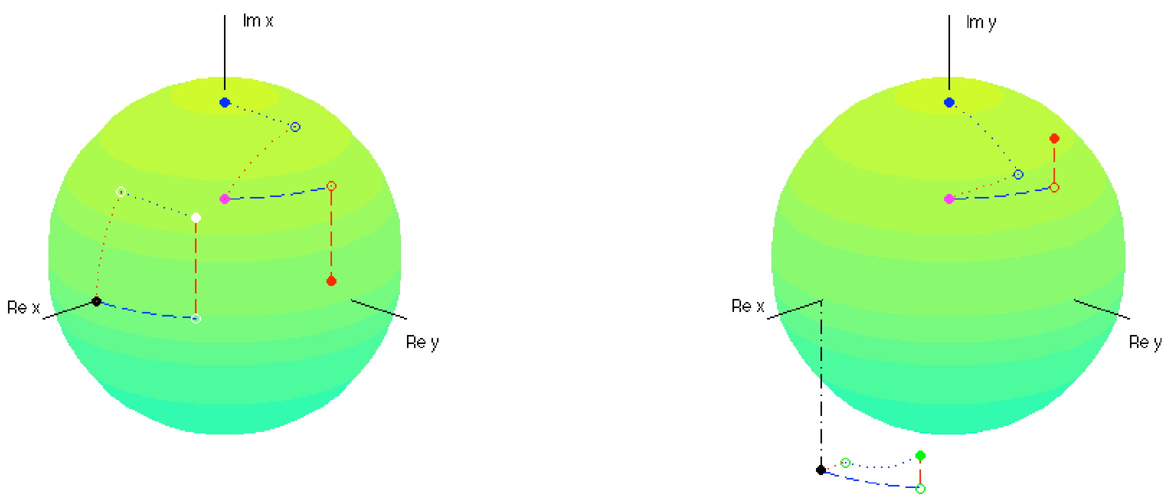


Figure 3